

GALOIS COHOMOLOGY OF REDUCTIVE ALGEBRAIC GROUPS OVER THE FIELD OF REAL NUMBERS

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ABSTRACT. We describe functorially the first Galois cohomology set $H^1(\mathbb{R}, G)$ of a connected reductive algebraic group G over the field \mathbb{R} of real numbers in terms of a certain action of the Weyl group on the real points of order dividing 2 of the maximal torus containing a maximal compact torus.

This result was announced with a sketch of proof in the author's 1988 note [4]. Here we give a detailed proof.

1. Introduction. Let G be a connected reductive algebraic group over the field \mathbb{R} of real numbers. We wish to compute the first Galois cohomology set $H^1(\mathbb{R}, G) = H^1(\text{Gal}(\mathbb{C}/\mathbb{R}), G(\mathbb{C}))$. In terms of Galois cohomology one can state answers to many natural questions, see e.g. Berhuy [2]. The Galois cohomology of classical groups and adjoint groups is well known. The Galois cohomology of compact groups was computed by Borel and Serre [3], Theorem 6.8, see also Serre [10], Section III.4.5. Here we consider the case of a general connected reductive group over \mathbb{R} . We describe $H^1(\mathbb{R}, G)$ in terms of a certain action of the Weyl group on the first Galois cohomology of the maximal torus containing a maximal compact torus. Our description is inspired by Borel and Serre [3]. Our result was announced in [4]; here we give a detailed proof. Note that Jeffrey Adams [1] computed, using in particular our result, the cardinalities of the first Galois cohomology sets of simple real groups.

The main result of this note is Theorem 9, which describes $H^1(\mathbb{R}, G)$.

2. We recall the definition of the first Galois cohomology set $H^1(\mathbb{R}, G)$ of an algebraic group G defined over \mathbb{R} . The set of 1-cocycles is defined by $Z^1(\mathbb{R}, G) = \{z \in G(\mathbb{C}) : z\bar{z} = 1\}$ where the bar denotes complex conjugation. The group $G(\mathbb{C})$ acts on the right on $Z^1(\mathbb{R}, G)$ by $z * x = x^{-1}z\bar{x}$, where $z \in Z^1(\mathbb{R}, G)$ and $x \in G(\mathbb{C})$. By definition $H^1(\mathbb{R}, G) = Z^1(\mathbb{R}, G)/G(\mathbb{C})$.

Let $G(\mathbb{R})_2$ denote the subset of elements of $G(\mathbb{R})$ of order 2 or 1. Then $G(\mathbb{R})_2 \subset Z^1(\mathbb{R}, G)$, and we obtain a canonical map $G(\mathbb{R})_2 \rightarrow H^1(\mathbb{R}, G)$.

2010 *Mathematics Subject Classification.* Primary: 11E72, 20G20.

Key words and phrases. Galois cohomology, real algebraic group, maximal torus.

The author was partially supported by the Hermann Minkowski Center for Geometry.

Lemma 3. *Let S be an algebraic \mathbb{R} -torus. Let S_0 denote the largest compact (i.e., anisotropic) \mathbb{R} -subtorus in S , and let S_1 denote the largest split subtorus in S . Then*

(a) *The map $\lambda: S(\mathbb{R})_2 \rightarrow H^1(\mathbb{R}, S)$ induces a canonical isomorphism $S(\mathbb{R})_2/S_1(\mathbb{R})_2 \xrightarrow{\sim} H^1(\mathbb{R}, S)$.*

(b) *The composite map $\mu: S_0(\mathbb{R})_2 \rightarrow H^1(\mathbb{R}, S_0) \rightarrow H^1(\mathbb{R}, S)$ is surjective.*

(c) *$(S_0 \cap S_1)(\mathbb{R}) = S_0(\mathbb{R})_2 \cap S_1(\mathbb{R})_2$, and the surjective map μ of (b) induces an isomorphism $S_0(\mathbb{R})_2/(S_0 \cap S_1)(\mathbb{R}) \xrightarrow{\sim} H^1(\mathbb{R}, S)$.*

Proof. Any \mathbb{R} -torus is isomorphic to a direct product of tori of three types: (1) $\mathbb{G}_{m,\mathbb{R}}$, (2) $R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}$, and (3) $R_{\mathbb{C}/\mathbb{R}}^1\mathbb{G}_{m,\mathbb{C}}$, cf. e.g. Voskresenskii [13], Section 10.1. Here \mathbb{G}_m denotes the multiplicative group, $R_{\mathbb{C}/\mathbb{R}}$ denotes the Weil restriction of scalars, and

$$R_{\mathbb{C}/\mathbb{R}}^1\mathbb{G}_{m,\mathbb{C}} = \ker[\mathrm{Nm}_{\mathbb{C}/\mathbb{R}}: R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}} \rightarrow \mathbb{G}_{m,\mathbb{R}}],$$

where $\mathrm{Nm}_{\mathbb{C}/\mathbb{R}}$ is the norm map.

We prove (a). The composite homomorphism $S_1(\mathbb{R})_2 \hookrightarrow S(\mathbb{R})_2 \rightarrow H^1(\mathbb{R}, S)$ factors via $H^1(\mathbb{R}, S_1) = 1$, and hence it is trivial. We obtain an induced homomorphism $S(\mathbb{R})_2/S_1(\mathbb{R})_2 \rightarrow H^1(\mathbb{R}, S)$; we must prove that it is bijective. It suffices to consider the three cases:

(1) $S = \mathbb{G}_{m,\mathbb{R}}$, i.e. $S(\mathbb{R}) = \mathbb{R}^\times$. Then $H^1(\mathbb{R}, S) = 1$. We have $S_1 = S$, so $S(\mathbb{R})_2/S_1(\mathbb{R})_2 = 1$. This proves (a) in case (1).

(2) $S = R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}$, i.e. $S(\mathbb{R}) = \mathbb{C}^\times$. Then $H^1(\mathbb{R}, S) = 1$. We have $S_1 = \mathbb{G}_{m,\mathbb{R}}$, $S_1(\mathbb{R}) = \mathbb{R}^\times$, $S_1(\mathbb{R})_2 = \{1, -1\} = S(\mathbb{R})_2$, so $S(\mathbb{R})_2/S_1(\mathbb{R})_2 = 1$. This proves (a) in case (2).

(3) $S = R_{\mathbb{C}/\mathbb{R}}^1\mathbb{G}_{m,\mathbb{C}}$, i.e. $S(\mathbb{R}) = \{x \in \mathbb{C}^\times : \mathrm{Nm}(x) = 1\}$, where $\mathrm{Nm}(x) = x\bar{x}$. Then by the definition of Galois cohomology $H^1(\mathbb{R}, S) = \mathbb{R}^\times/\mathrm{Nm}(\mathbb{C}^\times) \simeq \{-1, 1\}$. The homomorphism $S(\mathbb{R})_2 = \{-1, 1\} \rightarrow H^1(\mathbb{R}, S)$ is an isomorphism. This proves (a) in case (3).

Assertion (b) reduces to the cases (1), (2), (3), where it is obvious (note that only in case (3) we have $H^1(\mathbb{R}, S) \neq 1$).

Concerning (c), we have a commutative diagram

$$\begin{array}{ccc} S_0(\mathbb{R})_2 & \xrightarrow{\quad} & S(\mathbb{R})_2 \\ \downarrow & \searrow \mu & \downarrow \lambda \\ H^1(\mathbb{R}, S_0) & \longrightarrow & H^1(\mathbb{R}, S). \end{array}$$

We see from (a) that $\ker \mu = S_0(\mathbb{R})_2 \cap S_1(\mathbb{R})_2$, and we know from (b) that μ is surjective. Thus we obtain a canonical isomorphism

$$S_0(\mathbb{R})_2/(S_0(\mathbb{R})_2 \cap S_1(\mathbb{R})_2) \xrightarrow{\sim} H^1(\mathbb{R}, S).$$

It remains only to check that $S_0(\mathbb{R})_2 \cap S_1(\mathbb{R})_2 = (S_0 \cap S_1)(\mathbb{R})$. This can be easily checked in each of the cases (1), (2), (3) (note that only in case (2) this group is nontrivial). \square

Corollary 4. *Assume that S is an \mathbb{R} -torus such that $S = S' \times S''$, where S' is a compact torus and $S'' = R_{\mathbb{C}/\mathbb{R}}T$, where T is a \mathbb{C} -torus. Then $H^1(\mathbb{R}, S) = H^1(\mathbb{R}, S') = S'(\mathbb{R})_2$.*

Proof. The assertion follows from the proof of Lemma 3(a), because S' is a direct product of tori of type (3), hence $H^1(\mathbb{R}, S') = S'(\mathbb{R})_2$, and S'' is a direct product of tori of type (2), hence $H^1(\mathbb{R}, S'') = 1$. \square

We say that a connected real algebraic group H is *compact*, if the group $H(\mathbb{R})$ is compact, i.e. H is reductive and anisotropic.

We need the following two standard facts.

Lemma 5 (well-known). *Any nontrivial semisimple algebraic group H over \mathbb{R} contains a nontrivial connected compact subgroup.*

Proof. The assertion follows from the classification, (cf. e.g. Helgason [5], Section X.6.2, Table V). We prove it without using the classification. Let $\mathfrak{h} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition (see e.g. Onishchik and Vinberg [9], Section 4.3.1) of the Lie algebra $\mathfrak{h} = \text{Lie } H$. Then $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. If $\mathfrak{k} = 0$, then $\mathfrak{h} = \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] = 0$, hence \mathfrak{h} is commutative, which is clearly impossible. Thus $\mathfrak{k} \neq 0$. But \mathfrak{k} is the Lie algebra of a maximal compact subgroup K of H , hence H contains a nontrivial connected compact algebraic subgroup. \square

Lemma 6 (well-known). *Any two maximal compact tori in a connected reductive real algebraic group H are conjugate under $H(\mathbb{R})$.*

Proof. It suffices to prove that any two maximal compact tori in the derived group H^{der} of H are conjugate. This follows from the following well known facts from the theory of Lie groups: (1) Any two maximal compact subgroups in a connected semisimple Lie group are conjugate (cf. e.g. Onishchik and Vinberg [9], Ch. 4, Theorem 3.5); (2) Any two maximal tori in a connected compact Lie group are conjugate (cf. e.g. Onishchik and Vinberg [8], Section 5.2.7, Theorem 15). \square

7. Let G be a connected reductive algebraic group over \mathbb{R} . Let T_0 be a maximal compact torus in G . Set $T = \mathcal{Z}(T_0)$, $N_0 = \mathcal{N}(T_0)$, $W_0 = N_0/T$, where \mathcal{Z} and \mathcal{N} denote the centralizer and the normalizer in G , respectively.

We prove that T is a torus. By Humphreys [6], Theorem 22.3 and Corollary 26.2.A, the centralizer T of T_0 is a connected reductive \mathbb{R} -group. The torus T_0 is a maximal compact torus in T , and it is central in T . Since by Lemma 6 all the maximal compact tori in T are conjugate under $T(\mathbb{R})$, we see that T_0 is the only maximal compact torus in T . It follows that the derived group T^{der} of T contains no nontrivial compact tori. By Lemma 5 every nontrivial semisimple group over \mathbb{R} has a nontrivial compact connected algebraic subgroup, hence a nontrivial compact torus. We conclude that $T^{\text{der}} = 1$, hence T is a torus.

We have a right action of W_0 on T_0 defined by $(t, w) \mapsto t \cdot w := n^{-1}tn$, where $t \in T_0(\mathbb{C})$, $n \in N_0(\mathbb{C})$, n represents $w \in W_0(\mathbb{C})$. This action is defined over \mathbb{R} . We prove that $W_0(\mathbb{C})$ acts on T_0 effectively. Indeed, if $w \in W_0(\mathbb{C})$

with representative $n \in N_0(\mathbb{C})$ acts trivially on T_0 , then $n^{-1}tn = t$ for any $t \in T_0(\mathbb{C})$, hence $n \in T(\mathbb{C})$ (because the centralizer of T_0 is T), hence $w = 1$.

We prove that $W_0(\mathbb{C}) = W_0(\mathbb{R})$. We have seen that $W_0(\mathbb{C})$ embeds in $\text{Aut}_{\mathbb{C}}(T_0)$. Since T_0 is a compact torus, all the complex automorphisms of T_0 are defined over \mathbb{R} . We see that the complex conjugation acts trivially on $\text{Aut}_{\mathbb{C}}(T_0)$, and hence on $W_0(\mathbb{C})$. Thus $W_0(\mathbb{R}) = W_0(\mathbb{C})$.

Note that N_0 normalizes T , hence W_0 acts on T .

Construction 8. We define a right action of $W_0(\mathbb{R}) = W_0(\mathbb{C})$ on $H^1(\mathbb{R}, T)$. Let $z \in Z^1(\mathbb{R}, T)$, $n \in N_0(\mathbb{C})$, z represents $\xi \in H^1(\mathbb{R}, T)$, n represents $w \in W_0(\mathbb{R}) = W_0(\mathbb{C})$. We set

$$\xi * w = \text{Cl}(n^{-1}z\bar{n}) = \text{Cl}(n^{-1}zn \cdot n^{-1}\bar{n}),$$

where Cl denotes the cohomology class.

We prove that $*$ is a well defined action. First, since N_0 normalizes T and $z \in T(\mathbb{C})$, we see that $n^{-1}zn \in T(\mathbb{C})$. Now $w \in W_0(\mathbb{R})$, hence $w^{-1}\bar{w} = 1$ and $n^{-1}\bar{n} \in T(\mathbb{C})$. It follows that $n^{-1}z\bar{n} = n^{-1}zn \cdot n^{-1}\bar{n} \in T(\mathbb{C})$. We have

$$n^{-1}z\bar{n} \cdot \overline{n^{-1}z\bar{n}} = n^{-1}z\bar{n}\bar{n}^{-1}zn = 1$$

because $z\bar{z} = 1$. Thus $n^{-1}z\bar{n} \in Z^1(\mathbb{R}, T)$. If $z' \in Z^1(\mathbb{R}, T)$ is another representative of ξ , then $z' = t^{-1}z\bar{t}$ for some $t \in T(\mathbb{C})$, and

$$n^{-1}z'\bar{n} = n^{-1}t^{-1}z\bar{t}\bar{n} = (n^{-1}tn)^{-1} \cdot n^{-1}z\bar{n} \cdot \overline{n^{-1}tn} = (t')^{-1}(n^{-1}z\bar{n})\bar{t}'$$

where $t' = n^{-1}tn$, $t' \in T(\mathbb{C})$. We see that the cocycle $n^{-1}z'\bar{n} \in Z^1(\mathbb{R}, T)$ is cohomologous to $n^{-1}z\bar{n}$. If n' is another representative of w in $N_0(\mathbb{C})$, then $n' = nt$ for some $t \in T(\mathbb{C})$, and $(n')^{-1}z\bar{n}' = t^{-1}n^{-1}z\bar{n}\bar{t}$. We see that $(n')^{-1}z\bar{n}'$ is cohomologous to $n^{-1}z\bar{n}$. Thus $*$ is indeed a well defined action of the group $W_0(\mathbb{R})$ on the set $H^1(\mathbb{R}, T)$.

Note that in general $1 * w = \text{Cl}(n^{-1}\bar{n}) \neq 1$, and therefore, the action $*$ does not respect the group structure in $H^1(\mathbb{R}, T)$.

Let $\xi \in H^1(\mathbb{R}, T)$ and $w \in W_0(\mathbb{R})$. It follows from the definition of the action $*$ that the images of ξ and $\xi * w$ in $H^1(\mathbb{R}, G)$ are equal. We see that the map $H^1(\mathbb{R}, T) \rightarrow H^1(\mathbb{R}, G)$ induces a map $H^1(\mathbb{R}, T)/W_0(\mathbb{R}) \rightarrow H^1(\mathbb{R}, G)$.

Theorem 9. *Let G , T_0 , T , and W_0 be as above. The map*

$$H^1(\mathbb{R}, T)/W_0(\mathbb{R}) \rightarrow H^1(\mathbb{R}, G)$$

induced by the map $H^1(\mathbb{R}, T) \rightarrow H^1(\mathbb{R}, G)$ is a bijection.

Proof. We prove the surjectivity. It suffices to show that the map $H^1(\mathbb{R}, T) \rightarrow H^1(\mathbb{R}, G)$ is surjective. This was proved by Kottwitz [7], Lemma 10.2, with a reference to Shelstad [11]. We give a different proof. Let $\eta \in H^1(\mathbb{R}, G)$, $\eta = \text{Cl}(z)$, $z \in G(\mathbb{C})$, $z\bar{z} = 1$. Let $z = us = su$, where s and u are the semisimple and the unipotent parts of z , respectively (cf. Humphreys [6], Theorem 15.3). We have $us\bar{u}\bar{s} = 1$, where $\bar{u}\bar{s} = \bar{s}\bar{u}$ (because $us = su$). Thus $us = \bar{u}^{-1}\bar{s}^{-1}$, where u and \bar{u}^{-1} are unipotent, s and \bar{s}^{-1} are semisimple, $us = su$. From the equality $\bar{u}\bar{s} = \bar{s}\bar{u}$ it follows that $\bar{u}^{-1}\bar{s}^{-1} = \bar{s}^{-1}\bar{u}^{-1}$. Since

the Jordan decomposition in $G(\mathbb{C})$ is unique (cf. Humphreys [6], Theorem 15.3), we conclude that $s = \bar{s}^{-1}$, $u = \bar{u}^{-1}$. In other words, $s\bar{s} = 1$, $u\bar{u} = 1$, i.e. s and u are cocycles.

Since u is unipotent, the logarithm $\log(u) \in \text{Lie } G_{\mathbb{C}}$ is defined. We have $\log(u) + \overline{\log(u)} = 0$. Set $y = \frac{1}{2}\log(u)$, then $y + \bar{y} = 0$. We have $-y + \log(u) + \bar{y} = 0$, where $-y$, \bar{y} and $\log(u)$ pairwise commute. Set $u' = \exp(y)$, then $(u')^{-1}u\bar{u}' = 1$. Since s commutes with u , we have $\text{Ad}(s)y = y$, hence s commutes with u' . We obtain $(u')^{-1}su\bar{u}' = s$, hence the cocycle $z = su$ is cohomologous to the cocycle s , where s is semisimple.

We may and shall therefore assume that z is semisimple. Set $C = \mathcal{Z}_{G_{\mathbb{C}}}(z)$. Since $\bar{z} = z^{-1}$, we have $\overline{C} = C$, hence the algebraic subgroup C of $G_{\mathbb{C}}$ is defined over \mathbb{R} . The semisimple element z is contained in a maximal torus of $G_{\mathbb{C}}$ (cf. Humphreys [6], Theorem 22.2), hence z is contained in the identity component C^0 of C . The group C^0 is reductive, cf. Steinberg [12], Section 2.7(a). Let T' be a maximal torus of C^0 defined over \mathbb{R} , then $z \in T'(\mathbb{C})$, because z is contained in the center of C^0 . By Lemma 3(b) the class η of z comes from the maximal compact subtorus T'_0 of T' . By Lemma 6 any compact torus in G is conjugate under $G(\mathbb{R})$ to a subtorus of T_0 . Thus η comes from $H^1(\mathbb{R}, T_0)$, hence from $H^1(\mathbb{R}, T)$. This proves the surjectivity in Theorem 9.

We prove the injectivity in Theorem 9. Let $z, z' \in T(\mathbb{C})$, $z\bar{z} = 1$, $z'\bar{z}' = 1$, $z = x^{-1}z'\bar{x}$, where $x \in G(\mathbb{C})$. We shall prove that $z = n^{-1}z'\bar{n}$ for some $n \in N_0(\mathbb{C})$.

For $g \in G(\mathbb{C})$ set $g^\nu = z\bar{g}z^{-1}$. Then ν is an involutive antilinear automorphism of $G_{\mathbb{C}}$, and in this way we obtain a twisted form ${}_zG$ of G . Since $z \in T(\mathbb{C})$, the embeddings of the tori $T_{\mathbb{C}}$ and $T_{0,\mathbb{C}}$ into ${}_zG_{\mathbb{C}}$ are defined over \mathbb{R} . We denote the corresponding \mathbb{R} -tori of ${}_zG$ again by T and T_0 , respectively. The centralizer of T_0 in ${}_zG$ is T . The compact torus T_0 of ${}_zG$ is contained in some maximal compact torus S of ${}_zG$, and clearly S is contained in the centralizer T of T_0 in ${}_zG$. Since T_0 is the largest compact subtorus of T , we conclude that the $S = T_0$. Thus T_0 is a maximal compact torus in ${}_zG$.

Consider the embedding $i_x: t \mapsto x^{-1}tx: T_{0,\mathbb{C}} \rightarrow {}_zG_{\mathbb{C}}$. We have $i_x(t)^\nu = z\bar{x}^{-1}\bar{t}\bar{x}z^{-1}$. Since $z\bar{x}^{-1} = x^{-1}z'$, we obtain

$$z\bar{x}^{-1}\bar{t}\bar{x}z^{-1} = x^{-1}z'\bar{t}(z')^{-1}x = x^{-1}\bar{t}x = i_x(\bar{t}).$$

We see that $i_x(t)^\nu = i_x(\bar{t})$, hence i_x is defined over \mathbb{R} . Set $T'_0 = i_x(T_0)$; it is a compact algebraic torus in ${}_zG$, and $\dim T'_0 = \dim T_0$. Therefore T'_0 is conjugate to T_0 under ${}_zG(\mathbb{R})$, say $T_{0,\mathbb{C}} = h^{-1}T'_{0,\mathbb{C}}h$, where $h \in {}_zG(\mathbb{R})$. Set $n = xh$. Then $n^{-1}T_{0,\mathbb{C}}n = h^{-1}x^{-1}T_{0,\mathbb{C}}xh = h^{-1}T'_{0,\mathbb{C}}h = T_{0,\mathbb{C}}$, hence $n \in N_0(\mathbb{C})$. The condition $h \in {}_zG(\mathbb{R})$ means that $z\bar{h}z^{-1} = h$, or $h^{-1}z\bar{h} = z$. It follows that

$$n^{-1}z'\bar{n} = h^{-1}x^{-1}z'\bar{x}\bar{h} = h^{-1}z\bar{h} = z.$$

We have proved that there exists $n \in N_0(\mathbb{C})$ such that $z = n^{-1}z'\bar{n}$, i.e. $\text{Cl}(z)$ and $\text{Cl}(z')$ lie in the same orbit of $W_0(\mathbb{R})$ in $H^1(\mathbb{R}, T)$. This proves the injectivity in Theorem 9. \square

Remark 10. If G is a compact group, then Theorem 9 asserts that $H^1(\mathbb{R}, G) = T(\mathbb{R})_2/W$, where T is a maximal torus in G , and W is the Weyl group with the usual action. This was earlier proved by Borel and Serre [3].

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